ISRAEL JOURNAL OF MATHEMATICS 93 (1996), 399–428

CONFORMAL UNIFORMIZATION AND PACKINGS

BY

ODED SCHRAMM*

The Weizmann Institute of Science, Department of Theoretical Mathematics
Rehovot 76100, Israel
e-mail: schramm@wisdom.weizmann.ac.il

ABSTRACT

A new short proof is given for Brandt and Harrington's theorem about conformal uniformizations of planar finitely connected domains as domains with boundary components of specified shapes. This method of proof generalizes to periodic domains.

Letting the uniformized domains degenerate in a controlled manner, we deduce the existence of packings of specified shapes and with specified combinatorics. The shapes can be arbitrary smooth disks specified up to homothety, for example. The combinatorics of the packing is described by the contacts graph, which can be specified to be any finite planar graph whose vertices correspond to the shapes. This is in the spirit of Koebe's proof of the Circle Packing Theorem as a consequence of his uniformization by circle domains.

1. Introduction

CONFORMAL UNIFORMIZATION OF MULTIPLY CONNECTED DOMAINS. Riemann's mapping theorem tells us that every simply connected domain in the Riemann sphere $\hat{\mathbb{C}}$ is conformally homeomorphic to the unit disk U, to the plane $\mathbb{C}$, or to the sphere $\hat{\mathbb{C}}$. For domains that are not simply connected there is a more elaborate theory of conformal uniformization. Let D be some domain in the Riemann sphere $\hat{\mathbb{C}}$. It is a theorem that D is conformally homeomorphic to some horizontal slit domain; that is, a domain whose boundary components are all line segments parallel to the x-axis [9, V§2]. When D is finitely connected,

* The author thankfully acknowledges support of NSF grant DMS-9112150.
Received April 4, 1994

Koebe's uniformization theorem tells us that D is conformally homeomorphic to some circle domain D^*, a domain whose boundary components are circles and points. Moreover, D^* is unique up to Möbius transformations.

There are also various other such results (see [9], [24], [7], [12], [13]), but in the case of finite connectivity, the uniformization theorems of Brandt and Harrington are probably the most general and the ultimate. We defer the statement of these theorems to Section 3, and will state here their main corollary:

1.1 THEOREM [3], [11]: *Let D be a domain in $\hat{\mathbb{C}}$ with finitely many boundary components $B_1, \ldots, B_n$. Let $K_1, \ldots, K_n$ be compact connected sets in $\mathbb{C}$ such that $\mathbb{C} - K_j$ is connected and K_j contains more than a single point for each j. Then there are sets $K_1^*, \ldots, K_n^* \subset \mathbb{C}$, with each K_j^* being homothetic to K_j or a singleton, such that D is conformally homeomorphic to $D^* = \hat{\mathbb{C}} - \bigcup_{j=1}^n K_j^*$. Moreover, the conformal homeomorphism $f\colon D \to D^*$ can be required to match each B_j with ∂K_j^* and if $\infty \in D$ to satisfy the normalization*

$$\lim_{z \to \infty} f(z) - z = 0. \tag{1.1}$$

Recall that a homothety is a transformation of the form $z \to az + b$ with $a \in \mathbb{R}, a > 0$ and $b \in \mathbb{C}$. Two sets are **homothetic** if one is the image of the other under a homothety. The normalization (1.1) is equivalent to the requirement that f has an expansion of the form $f(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$ near infinity.

When one sets $K_j = [0,1], j = 1, \ldots, n$ in 1.1, one gets uniformization by horizontal slit domains in the finitely connected case, and when one sets each K_j to be a disk, Koebe's theorem is obtained. Courant, Manel and Shiffman [7] have proved Theorem 1.1 in the special case that all the sets K_j are convex.

In this paper we present a new short proof of Harrington's uniformization theorem. The proof is based on the topological notion of the degree of a proper map between manifolds. An advantage of this proof is that it allows one to replace (1.1) by various other natural normalizations, and generalizes to periodic domains, as we do in Sections 4 and 5. Brandt's theorem came to the attention of the author only after the completion of a few drafts of this work. It is very similar to Harrington's theorem. Brandt's proof is also short, elegant and permits variations of the normalization (1.1).

In general, the normalization (1.1) or its variants do not make the uniformized domain D^* unique. (We discuss uniqueness issues below.)

PACKINGS. For us, a **packing** is an indexed collection $P = (P_v: v \in V)$ of compact connected sets in the Riemann sphere $\hat{\mathbb{C}}$ such that the interior of each set is disjoint from the other sets. The **contacts graph** of the packing is a combinatorial graph $G = (V, E)$, whose vertex set is the indexing set of the packing and where an edge $[u, w]$ appears in E precisely when $P_u \cap P_w \neq \emptyset$. In the literature, the terms 'nerve' and 'tangency graph' are also used. It is easy to show that the contacts graph is a planar graph when the sets in the packing P are smooth disks. We now state the remarkable

1.2 CIRCLE PACKING THEOREM [15]: *Let $G = (V, E)$ be a finite planar graph. Then there exists a packing of geometric disks in the plane $P = (P_v: v \in V)$ with contacts graph G. Furthermore, if G is isomorphic to the 1-skeleton of a triangulation of the Riemann sphere $\hat{\mathbb{C}}$, then P is unique up to Möbius transformations.*

Note that here and in the following the graphs are assumed to be without multiple edges and every edge has two distinct vertices.

This theorem was first discovered by Koebe as a corollary to his conformal uniformization by circle domains. His argument can probably be sketched as follows. Starting with the graph G, one constructs a domain D in the plane whose boundary components are in one-to-one correspondence with the vertices of G. Without much difficulty, one arranges so that for every edge $[u, w]$ of G the corresponding boundary components 'almost' touch. Then the uniformization theorem says that there is a circle domain D^* that is conformally homeomorphic to D. One argues that also in D^* circles that correspond to neighboring vertices are 'almost' tangent. In other words, the connected components of $\hat{\mathbb{C}} - D^*$ are disks, which 'nearly' have the contacts dictated by G. Then a limiting argument completes the proof. (One does not need to worry about excessive contacts.)

In Section 6 we give a careful proof along these lines, but instead of using Koebe's uniformization by circle domains we use Theorem 1.1 and its variants. The result is a much more general packing theorem, in which the shapes of the packed sets can be arbitrary smooth disks. See Theorems 6.1 and 6.3. We close Section 6 by proving a packing theorem for doubly periodic graphs.

We have mentioned that the mapping f in Theorem 3.1 is not unique, even if one imposes the normalization (1.1). This is shown in Section 7. However, Shiffman has shown that f is uniquely determined when the sets K_j are strictly convex [23]. We adapt Shiffman's method to packings, and prove a uniqueness theorem for packings of convex shapes.

Recently, the Circle Packing Theorem has been rediscovered by Thurston ([26], [25, Chapter 13]) as a corollary of Andreev's Theorem ([1], [2]) about the realizability of hyperbolic polyhedra with prescribed angles. Thurston gave a new proof for the theorem, and generalized the theorem to closed hyperbolic surfaces, and to patterns of circles with prescribed angles of intersection. He also conjectured that circle packings can yield approximations to conformal maps; that conjecture was later proved by Rodin and Sullivan [18]. Since then, interesting new proofs were given to the Circle Packing Theorem. A proof based on a convexity argument can be found in [6], and a Perron type approach is given in [5], [4]. In [19] the key is an application of Brouwer's fixed point theorem, and the theorem is generalized to convex sets. Transversality is the main point in [22], and uniqueness based on separation arguments in the plane is the theme of [20], which also applies to the convex case.

2. Notations and definitions

A domain in $\hat{\mathbb{C}}$ is a connected open set contained in $\hat{\mathbb{C}}$.

Let $\mathcal{D}^n$ be the space of all domains $D \subset \hat{\mathbb{C}}$ that have exactly n boundary components and these have labels $1, 2, \ldots, n$. (This means that, technically speaking, a point in $\mathcal{D}^n$ consists of a pair, (D, l) where $D \subset \hat{\mathbb{C}}$ is a domain and l is a bijection from $\{1, 2, \ldots, n\}$ to the set of boundary components of D.) Elements of $\mathcal{D}^n$ will be called **labeled domains**. When, $D \in \mathcal{D}^n$ we denote the j-th boundary component of D by $\partial_j D$. Given $D_1, D_2 \in \mathcal{D}^n$, we say that $f\colon D_1 \to D_2$ is a **label preserving conformal homeomorphism** if it is a conformal homeomorphism between the domains and it respects the labeling; that is, for each $j = 1, \ldots, n$, $\partial_j D_2$ is the boundary component of D_2 that corresponds under f to $\partial_j D_1$. (Every homeomorphism between domains in $\hat{\mathbb{C}}$ induces a bijection between the sets of boundary components.) Henceforth, the homeomorphisms under consideration will always be label preserving, and thus the phrase "label preserving" will sometimes be omitted.

Given $D \in \mathcal{D}^n$ and $j = 1, 2, \ldots, n$, there is a unique connected component of $\hat{\mathbb{C}} - D$ that contains $\partial_j D$. We call it the j-th **hole** of D, and denote it by $\hat{\partial}_j D$. In general, a **hole** H is a compact subset of $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} - H$ is homeomorphic to a disk. A **nondegenerate hole** is a hole that contains more than a single point, and a **bounded hole** is a hole that does not contain ∞. It is clear that $\hat{\partial}_j D$ is a hole when $D \in \mathcal{D}^n$, $j = 1, 2, \ldots, n$, and so the definitions are consistent. An equivalent definition for a hole is 'a compact connected nonempty proper subset of $\hat{\mathbb{C}}$ whose complement is connected'.

Define $\mathcal{D}^n_{(\infty)} = \{D \in \mathcal{D}^n: \infty \in D\}$. The most natural topology on $\mathcal{D}^n_{(\infty)}$ may seem to be the topology induced by Hausdorff convergence of each $\hat{\partial}_j D$, but this is not the appropriate topology for our purposes, since the Hausdorff limit of bounded holes may fail to be a hole. For example, the Hausdorff limit as $t \to 1$ of arcs of length $2\pi t$ on the unit circle is the whole unit circle, which is not a hole.

We shall now define a metric on $\mathcal{D}^n_{(\infty)}$. Denote the collection of all bounded holes by $\mathcal{H}$, and let $H_1, H_2 \in \mathcal{H}$. Define $\rho_*(H_1, H_2)$ as the least $\rho \geqslant 0$ such that H_2 is disjoint from the unbounded component of $\mathbb{C} - \overline{N}_\rho(H_1)$, where $\overline{N}_\rho(H_1) = \{z \in \mathbb{C}: d(z, H_1) \leqslant \rho\}$ and $d(z, H_1)$ is the distance from z to H_1. Now set

$$\rho(H_1, H_2) = \max\{\rho_*(H_1, H_2), \rho_*(H_2, H_1)\}.$$

This is easily seen to be a metric on $\mathcal{H}$. If a sequence $H_1, H_2, \ldots$ in $\mathcal{H}$ has a Hausdorff limit $K \subset \mathbb{C}$, then the ρ-limit of this sequence is the complement of the unbounded connected component of $\mathbb{C} - K$. In particular, from the usual compactness property of the Hausdorff metric we see that the set of holes contained in a compact subset of $\mathbb{C}$ is compact. The metric on $\mathcal{D}^n_{(\infty)}$ is now defined by:

$$d(D_1, D_2) = \sum_{j=1}^{n} \rho(\hat{\partial}_j D_1, \hat{\partial}_j D_2)$$

for $D_1, D_2 \in \mathcal{D}^n_{(\infty)}$. To correlate this metric with the established terminology we note that convergence in $\mathcal{D}^n_{(\infty)}$ is equivalent to kernel convergence together with 'consistency of the labels'.

Given a positive integer n and a set A, we denote by $A^n_{\neq}$ the set of points $(a_1, \ldots, a_n) \in A^n$ that satisfy $a_j \neq a_k$ for all $j \neq k$, $j, k = 1, 2, \ldots, n$. When $K \subset \mathbb{C}$, $a \in \mathbb{R}$, we denote the set $\{az: z \in K\}$ by aK. Similarly, $K + c = \{z + c: z \in K\}$.

We also set $\mathbb{R}_+ = \{r \in \mathbb{R}: r \geqslant 0\}$.

3. The finite connectivity uniformization theorems

We recall that a map $\phi: A \to B$ is **proper** if the inverse image of every compact set in B is compact.

3.1 HARRINGTON'S THEOREM[11]: *Let P be a relatively open subset of $\mathbb{C}^n \times \mathbb{R}_+^n$ containing $\mathbb{C}_{\neq}^n \times \{0\}^n$. Let ϕ: $P \to \mathcal{D}_{(\infty)}^n$ be a proper continuous function. Suppose that*

$$\partial_j \phi(c,r) = \{c_j\} \quad \textit{whenever } (c,r) \in P \textit{ and } r_j = 0 \tag{3.1}$$

holds for $j = 1, 2 \ldots, n$. ($\partial_j \phi(c,r)$ denotes the j-th boundary component of $\phi(c,r)$.) Then every domain $A \in \mathcal{D}_{(\infty)}^n$ is conformally homeomorphic to some domain A^* in the image of ϕ, $A^* \in \phi(P)$. Furthermore, the conformal homeomorphism f: $A \to A^*$ can be required to be label preserving and to satisfy the normalization (1.1).

This statement of the theorem is slightly stronger than the one appearing in [11], because the topology we use for $\mathcal{D}_{(\infty)}^n$ is weaker.

Theorem 1.1 is a direct corollary of Theorem 3.1. We now repeat the simple argument of [11] to that effect.

Proof of Theorem 1.1 using Theorem 3.1: We assume without loss of generality that $0 \in K_j$ for each j. Let $K_j(c,r) = r_j K_j + c_j$ for $(c,r) \in \mathbb{C}^n \times \mathbb{R}_+^n$, let P be the set of (c,r) such that $K_j(c,r) \cap K_m(c,r) = \emptyset$ for $j \neq m$, and for $(c,r) \in P$ let $\phi(c,r)$ be the domain in $\mathcal{D}_{(\infty)}^n$ whose complementary components are $K_j(c,r)$, $j = 1, \ldots, n$. Theorem 3.1 now clearly follows from 1.1. ∎

3.2 Remark: Of course, Theorem 3.1 can give much more general uniformizations. For $j = 1, 2, \ldots, n$ let K_j: $\mathbb{C} \times \mathbb{R}_+ \to \mathcal{H}$ be continuous and proper, and suppose that $K_j(c,0) = \{c\}$ for every $c \in \mathbb{C}$. Then Theorem 1.1 holds when one requires each K_j^* to be equal to some $K_j(c,r)$, instead of specifying it up to homothety. The proof is the same.

For example, we can require K_1^* to be a line segment whose direction is e^{ir}, where r is its length, require K_2^* to be an ellipse whose eccentricity and inclination depend continuously on the position of the center, and require K_3^* to be a circular arc whose center is $f(z_1)$, where z_1 is some arbitrarily chosen point in D and f is the conformal homeomorphism. (This last specification does not fall exactly into the framework of the previous paragraph, but does follow from 3.1. The trick needed is to let $\{z_1\}$ be an artificially introduced additional boundary component.)

A further example is the following. Let d be some Riemannian metric on $\mathbb{C}$. For each compact ball in this metric $B_d(c,r)$ with center c and radius r let $b(B_d(c,r))$

be the complement of the unbounded connected component of $\mathbb{C} - B(c,r)$. (We have $b(B_d(c,r)) = B_d(c,r)$ if $B_d(c,r)$ is simply connected.) Then every domain in $\mathcal{D}^n_{(\infty)}$ is conformally homeomorphic to a domain of the form $\hat{\mathbb{C}} - \bigcup_{j=1}^n b(B_d(c_j,r_j))$ for an appropriate choice of c_j, r_j (allowing also $r_j = 0$), and the conformal homeomorphism can be required to satisfy the normalization (1.1).

A similar remark applies to the uniformization and packing theorems below.

Theorem 1.1 is also an easy corollary of

3.3 Brandt's Theorem[3]: *Let n be a positive integer, let $D \in \mathcal{D}^n_{(\infty)}$, and let Σ denote the space of all conformal homeomorphisms from D into $\hat{\mathbb{C}}$ that satisfy the normalization* (1.1). *Suppose that for each $j = 1, \dots, n$ a continuous map*

$$A_j \colon \mathcal{H} \times \Sigma \to \mathcal{H}$$

is given. Further suppose that for every $j = 1, \dots, n$ and for every $(H, f) \in \mathcal{H} \times \Sigma$ there exists a conformal homeomorphism satisfying (1.1) *from $\hat{\mathbb{C}} - H$ onto $\hat{\mathbb{C}} - A_j(H, f)$. Then there is an $f \in \Sigma$ such that*

$$\hat{\partial}_j f(D) = A_j(\hat{\partial}_j f(D), f)$$

holds for $j = 1, \dots, n$.

Theorems 3.3 and 3.1 are very similar. Both provide the existence of a conformal map $f \colon D \to \hat{\mathbb{C}}$ satisfying certain conditions. In Brandt's theorem f satisfies a system of equations, while in Harrington's theorem $f(D)$ is in the image of a map.

Proof of Harrington's Theorem 3.1: Let $D \in \mathcal{D}^n_{(\infty)}$. There is a unique conformal homeomorphism g_D from D onto a horizontal slit domain that satisfies the normalization (1.1). (It is a consequence of the Area Theorem that the (injective) conformal map that satisfies the normalization and maximizes $\lim_{z\to\infty} \operatorname{Re} z(g(z) - z)$ is this g_D [9, Chapter V, §2].) Let $\mathcal{S} \subset \mathcal{D}^n_{(\infty)}$ be the collection of those domains whose boundary components are all horizontal slits, and let $\psi(D) = g_D(D)$.

We will now show that $\psi \colon \mathcal{D}^n_{(\infty)} \to \mathcal{S}$ is continuous. Let $D_1, D_2, \dots$ be a sequence in $\mathcal{D}^n_{(\infty)}$ that converges to $D \in \mathcal{D}^n_{(\infty)}$. From the normalization of the mappings g_{D_k}, $k = 1, 2, \dots$, it follows that this sequence forms a normal family on compact subsets of D. Thus a subsequence, $(g_{D_k} \colon k \in J)$, converges uniformly on compact subsets of D, say to g. It is easy to see that g must also satisfy the normalization (1.1), and is therefore not a constant. Since a locally uniform limit of conformal maps is either a constant or is conformal, g is conformal. Now, Carathéodory's Kernel Convergence Theorem [9, V§5, Theorem 1] implies that the domains $(g_{D_k}(D_k)\colon k \in J)$ converge to $g(D)$; that is, $(\psi(D_k)\colon k \in J)$ converges to $\psi(D)$. This proves the continuity of ψ.

We now establish that ψ is proper. Given $W \in \mathcal{D}^n_{(\infty)}$, let $U(W)$ be the set of all $S \in \mathcal{S}$ such that $\hat{\partial}_j S$ is contained in the interior of $\hat{\partial}_j W$ for each $j = 1, \ldots, n$. This is an open set in $\mathcal{S}$. Let K be a compact subset of $\mathcal{S}$, and let $S \in K$. There clearly is some $W_S \in \mathcal{D}^n_{(\infty)}$ such that $S \in U(W_S)$. As in the above paragraph, the collection $\mathcal{T}$ of all conformal maps of W_S that satisfy the normalization (1.1) is compact under the topology of uniform convergence on compact subsets of W_S. Therefore, there is some $\epsilon > 0$ so that for every $h \in \mathcal{T}$ all the boundary components of $D = h(W_S)$ are contained in the disk of radius $1/\epsilon$ around 0 and $\hat{\partial}_j D$ is contained in the unbounded component of $\mathbb{C} - N_\epsilon(\hat{\partial}_k D)$, for any two distinct $j, k \in \{1, \ldots, n\}$. Here $N_\epsilon(\hat{\partial}_k D)$ denotes the ϵ-neighborhood of $\hat{\partial}_k D$. A conformal map satisfies the normalization (1.1) if its inverse does, and so the restriction of g_D^{-1} to W_S is in $\mathcal{T}$ whenever $D \in \psi^{-1}(U(W_S))$. Therefore, every $D \in \psi^{-1}(U(W_S))$ satisfies the above conditions with ϵ. This implies that $\psi^{-1}(U(W_S))$ has compact closure in $\mathcal{D}^n_{(\infty)}$. Since, by compactness, finitely many sets of the form $U(W)$ cover K, we see that $\psi^{-1}(K)$ is contained in a compact subset of $\mathcal{D}^n_{(\infty)}$. But $\psi^{-1}(K)$ is closed, since ψ is continuous. Therefore, $\psi^{-1}(K)$ is compact, and ψ is proper.

Denote $\eta = \psi \circ \phi\colon P \to \mathcal{S}$. We will prove that η is surjective. Let us first introduce a convenient coordinate system on $\mathcal{S}$. Take any

$$(c, r) = (c_1, \ldots, c_n, r_1, \ldots, r_n) \in \mathbb{C}^n \times \mathbb{R}^n_+.$$

Consider the n horizontal slits whose left endpoints are $c_1, \ldots, c_n$ and whose lengths are $r_1, \ldots, r_n$, respectively. Let Q be the set of such points (c, r) so that the slits are disjoint, and when $(c, r) \in Q$ let $S(c, r) \in \mathcal{S}$ be the corresponding slit domain.

With this coordinate system on $\mathcal{S}$, η can be thought of as a map from $P \subset \mathbb{C}^n \times \mathbb{R}^n_+$ to $Q \subset \mathbb{C}^n \times \mathbb{R}^n_+$. It is clear that η is continuous and proper. The following lemma will show that η is also surjective.

3.4 Degree Lemma: *Let M be a topological manifold, let P, Q be relatively open connected subsets of $M \times \mathbb{R}^n_+$, and let*

$$M_0 = \{m \in M: (m, 0) \in P\}.$$

*Suppose that η: $P \to Q$ is a continuous proper map such that $\eta(m, 0) = (m, 0)$ whenever $m \in M_0$. Also suppose that for each $j = 1, \dots, n$ we have $r^*_j = 0$ if $\eta(m, r) = (m^*, r^*)$ and $r_j = 0$. That is, η is the identity on $M_0 \times \{0\}^n$, and $\eta(P \cap (M \times \mathbb{R}^{j-1}_+ \times \{0\} \times \mathbb{R}^{n-j}_+)) \subset M \times \mathbb{R}^{j-1}_+ \times \{0\} \times \mathbb{R}^{n-j}_+$. Further assume that $M_0 \neq \emptyset$. Then the degree of η is one, and thus η is surjective.**

See [8, VIII.4] regarding the notion of the degree of a proper map between manifolds. A reader more comfortable in the differentiable setting can avoid the use of topological degree by first smoothing the map $\tilde{\eta}$ below, and then using the properties of the differentiable degree ([17], [10]), which are a consequence of Sard's Theorem, to establish that the smoothed map is surjective.

Proof: Given $a \in \mathbb{R}$, let $a^+ = \max(a, 0)$, and let $r^+ = (r^+_1, \dots, r^+_n)$ for $r = (r_1, \dots, r_n) \in \mathbb{R}^n$. Let $\tilde{P}, \tilde{Q}$ be the inverse images of P and Q, respectively, under the map $\mathrm{id} \times + : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ that takes each (m, r) to (m, r^+). Extend η to a map $\tilde{\eta}$: $\tilde{P} \to \tilde{Q}$, $\tilde{\eta}(m, r) = (\tilde{m}(m, r), \tilde{r}(m, r))$, defined by

$$\begin{aligned} (\tilde{m}(m, r), \tilde{r}(m, r)^+) &= \eta(m, r^+), \\ \tilde{r}_j(m, r) &\geqslant 0 \quad \text{whenever } r_j \geqslant 0, \\ \tilde{r}_j(m, r) &= r_j \quad \text{whenever } r_j < 0. \end{aligned}$$

It is clear that $\tilde{\eta}$ is a proper continuous map from $\tilde{P}$ to $\tilde{Q}$ and that these are open and connected subsets of $M \times \mathbb{R}^n$. Therefore, $\tilde{\eta}$ has a well defined degree. To see what that degree is, we look at the negative orthant of $\tilde{Q}$, $\tilde{Q}_- = \{(m, r) \in \tilde{Q}: r_j < 0 \text{ for } j = 1, \dots, n\}$. The preimage of $\tilde{Q}_-$ is the negative orthant of $\tilde{P}$, and $\tilde{\eta}(m, r) = (m, r)$ there. Thus the degree of $\tilde{\eta}$ is 1, and the same is true for η. This proves the lemma. ∎

Proof of Theorem 3.1, Continued: Take $M = \mathbb{C}^n$. We assume without loss of generality that P is connected, since otherwise we may replace P by the connected component of it that contains $\mathbb{C}^n_{\neq} \times \{0\}^n$. To show that $\eta(P) = Q$ we only need

* If M is not orientable, then the degree of η is not necessarily defined as an integer, but only mod 2.

to verify the remaining hypotheses of the Degree Lemma, and these follow from (3.1) and the removablity of isolated singularities for bounded analytic functions. Given $A \in \mathcal{D}^n_{(\infty)}$, let $p \in \eta^{-1}(\{\psi(A)\})$. Then A is conformally homeomorphic to $\phi(p)$, and the conformal homeomorphism, $g^{-1}_{\phi(p)} \circ g_A$, satisfies the normalization (1.1). ∎

Remark: The proof that ψ is proper shows that the following statement holds. Let K be a compact subset of $\mathcal{D}^n_{(\infty)}$, and let $\tilde{K}$ be the collection of all $D \in \mathcal{D}^n_{(\infty)}$ such that there is some $D' \in K$ and a conformal homeomorphism $h: D \to D'$ that satisfies the normalization (1.1). Then $\tilde{K}$ is compact.

4. Other normalizations for the uniformizing map

One of the classical uniformizations for finitely connected domains is by means of circle domains, domains with circle and point boundary components. In that setting, the uniformization is unique up to Möbius transformations. In order to get a unique uniformization, one can impose some normalization. A natural normalization is to fix the images of three distinct points of the domain. Another possibility is to fix the value and the first two derivatives at a given point. (This can be seen as a limiting case for fixing the values at three distinct points, when the points get closer and closer.) Yet a third possibility is fixing one nontrivial boundary component to be a given circle, say the unit circle, and fixing three points on it. And there are still other variations. As one generalizes circle uniformizations to more general uniformizations, it is of interest to see which of these normalizations can be made to work.

The normalization (1.1) corresponds to fixing the value and the first two derivatives at a point, where the point and its image are chosen to be ∞. The goal of this section is to show that our method of proof of Theorem 3.1 also gives uniformizations with other normalizations.

4.1 Theorem: *Let $A \in \mathcal{D}^n$, let $w_0, w_1, w_2 \in A$ be three distinct points with $\infty \neq w_0, \infty \neq w_1$, and let $K_1, \dots, K_n \in \mathcal{H}$ be nondegenerate holes that contain 0. Then there is a (label preserving) conformal homeomorphism $f: A \to A^*$, such that each $\hat{\partial}_j A^*$ is homothetic to K_j or consists of a single point for $j = 1, \dots, n$. Moreover, any one of the following normalizations can be imposed.*

(1) $f(w_0) = 0, f(w_1) = 1$ *and* $f(w_2) = \infty$.
(2) $f(w_0) = 0, f'(w_1) = 1$ *and* $f(w_2) = \infty$.

(3) *$f(w_0) = 0, f'(w_0) = 1$ and $f(w_2) = \infty$.*
(4) *If $n > 0$, then $\hat{\partial}_1 A^*$ has the form $r_1 K_1$ for some $r_1 \geqslant 0$, and $f(w_1) = 1, f(w_2) = \infty$.*
(5) *If $n > 0$, then $\hat{\partial}_1 A^*$ has the form $r_1 K_1$ for some $r_1 \geqslant 0$, and $f'(w_1) = 1, f(w_2) = \infty$.*
(6) *If $n > 1$, then $\hat{\partial}_1 A^*$ has the form $r_1 K_1$, $\hat{\partial}_2 A^*$ has the form $r_2 K_2 + 1$, and $f(w_2) = \infty$. Here $r_1, r_2 \in \mathbb{R}_+$.*

4.2 THEOREM: *Let $A \in \mathcal{D}^n$, and suppose that $\partial_n A$ is a Jordan curve. Let $q_0, q_1, q_2 \in \partial_n A$ be three distinct points in positive order, and let $w_0, w_1 \in A$. Let J be a Jordan domain in $\mathbb{C}$, let p_0, p_1, p_2 be three distinct points on ∂J in positive order, and let γ: $[0,1) \to J$ be proper and continuous. Suppose that $K_1, \ldots, K_{n-1} \in \mathcal{H}$ are nondegenerate holes that contain 0. Then there is a conformal homeomorphism $f: A \to A^*$, such that $A^* \subset J$, $\partial_n A^* = \partial J$, and each $\hat{\partial}_j A^*$ is homothetic to K_j or consists of a single point for $j = 1, 2, \ldots, n-1$. Moreover, any one of the following normalizations can be imposed.*

(1) *$f(q_j) = p_j$ holds for each j=0, 1, 2.*
(2) *$f(w_0) = \gamma(0)$ and $f'(w_0) > 0$.*
(3) *$f(w_0) = \gamma(0)$ and $f(q_0) = p_0$.*
(4) *$f(w_0) = \gamma(0)$ and $f(w_1) = \gamma(t)$ for some $t \in [0,1)$.*
(5) *If $n > 2$, then $\hat{\partial}_1 A^*$ has the form $r_1 K_1 + \gamma(0)$ for some $r_1 \geqslant 0$, and $\hat{\partial}_2 A^*$ has the form $r_2 K_2 + \gamma(t)$ for some $r_2 \geqslant 0$, $t \in (0,1)$.*
(6) *If $n > 1$, then $\hat{\partial}_1 A^*$ has the form $r_1 K_1 + \gamma(0)$ for some $r_1 \geqslant 0$, and $f(q_0) = p_0$.*
(7) *If $n > 1$, then $f(w_0) = \gamma(0)$, and $\hat{\partial}_1 A^*$ has the form $r_1 K_1 + \gamma(t)$ for some $r_1 \geqslant 0$, $t \in (0,1)$.*

In some of these normalizations we have used f to denote also its continuous extension to $\partial_n A$.

The proofs for all these normalizations are very similar, and are similar to the proof of 1.1. We will only give the proof of (5) in Theorem 4.2. Note that normalizations (4) and (7) follow from normalization (5), since w_0 and w_1 can be deleted from A to introduce additional boundary components, and normalizations (2), (3) and (6) can then be obtained through a limiting process.

We now introduce some additional notations. Let J be as in Theorem 4.2, and denote by $\mathcal{D}^n_J$ the set of all domains $D \in \mathcal{D}^n$ that are contained in J and whose

n-th boundary component is ∂J; that is, $\partial_n D = \partial J$. Define the metric on $\mathcal{D}^n_J$ to be $d(D_1, D_2) = \sum_{j=1}^{n-1} \rho(\hat{\partial}_j D_1, \hat{\partial}_j D_2)$, where ρ is as in Section 2.

The following theorem is analogous to Theorem 3.1.

4.3 THEOREM: *Let J and γ be as in Theorem 4.2. Let $n \geqslant 3$, and let P be a relatively open subset of $J^{n-3} \times [0,1) \times \mathbb{R}^{n-1}_+$ that intersects $J^{n-3} \times [0,1) \times \{0\}^{n-1}$ Let $\phi: P \to \mathcal{D}^n_J$ be a proper continuous function. Suppose that*

$$\partial_j \phi(c,t,r) = \{c_j\} \quad \text{whenever } (c,t,r) \in P \text{ and } r_j = 0 \tag{4.1}$$

holds for $j = 1, 2 \ldots, n-3$, and

$$\partial_{n-2}\phi(c,t,r) = \{\gamma(t)\} \quad \text{whenever } (c,t,r) \in P \text{ and } r_{n-2} = 0,$$
$$\partial_{n-1}\phi(c,t,r) = \{\gamma(0)\} \quad \text{whenever } (c,t,r) \in P \text{ and } r_{n-1} = 0,$$
$$\gamma(0) \in \hat{\partial}_{n-1}\phi(c,t,r) \quad \text{whenever } (c,t,r) \in P.$$

Then every domain $A \in \mathcal{D}^n_J$ is conformally homeomorphic to some domain A^* in the image of ϕ, $A^* \in \phi(P)$.

Proof: Let P_0 be a connected component of P that intersects $J^{n-3} \times [0,1) \times \{0\}^{n-1}$, and let ϕ_0 be the restriction of ϕ to P. Let $\mathcal{D}^{n*}_J$ be the collection of all domains $D \in \mathcal{D}^n_J$ such that $\gamma(0) \in \hat{\partial}_{n-1} D$. By our assumptions, the image of ϕ is contained in $\mathcal{D}^{n*}_J$.

Let $\mathcal{C}^n$ be the collection of all domains $D \in \mathcal{D}^n$ such that all the boundary components of D are circles or points, such that $\partial_n D = \partial U$ (the unit circle), such that 0 is the center of $\partial_{n-1} D$, and such that the center of $\partial_{n-2} D$ lies on the positive real ray $\mathbb{R}_+$.

Given any $D \in \mathcal{D}^{n*}_J$ there is a unique conformal homeomorphism $g_D: D \to D^*$, such that $D^* \in \mathcal{C}^n$. (Existence follows from Theorem 1.1 using Möbius transformations. For a discussion of uniqueness see Section 7. In this case of circles, both uniqueness and existence are classical, see [14], [9].) Let $\psi: \mathcal{D}^{n*}_J \to \mathcal{C}^n$ be defined by $\psi(D) = g_D(D)$. Then, as in the proof of Theorem 3.1, it is not hard to show that ψ is continuous and proper.

We'll prove that $\psi \circ \phi_0: P_0 \to \mathcal{C}^n$ is surjective, and to this end we introduce the following parameterization of $\mathcal{C}^n$. Let $x = (c,t,r) \in U^{n-3} \times [0,1) \times \mathbb{R}^{n-1}_+$. For $j = 1, \ldots, n-3$, let $C_j(x)$ be the circle with center c_j and radius r_j, let $C_{n-2}(x)$ be the circle with center t and radius r_{n-2}, let $C_{n-1}(x)$ be the circle with center 0 and radius r_{n-1}, and let $C_n(x) = \partial U$. Let Q be the collection of those $x \in J^{n-3} \times [0,1) \times \mathbb{R}_+^{n-1}$ such that there is a domain $C(x) \in \mathcal{C}^n$ with $\partial_j C(x) = C_j(x)$ for $j = 1, \ldots, n$. Then $C\colon Q \to \mathcal{C}^n$ is a homeomorphism.

Now denote $\eta = C^{-1} \circ \psi \circ \phi\colon P_0 \to Q$; this is map from an open subset of $J^{n-3} \times [0,1) \times \mathbb{R}_+^{n-1}$ to an open subset of $U^{n-3} \times [0,1) \times \mathbb{R}_+^{n-1}$. It is clear that η is continuous and proper. Moreover, $\eta(P_0 \cap J^{n-3} \times [0,1) \times \mathbb{R}_+^{j-1} \times \{0\} \times \mathbb{R}_+^{n-j-1}) \subset U^{n-3} \times [0,1) \times \mathbb{R}_+^{j-1} \times \{0\} \times \mathbb{R}_+^{n-j-1}$ holds for $j = 1, \ldots, n-1$. Since P_0 intersects $J^{n-3} \times [0,1) \times \{0\}^{n-1}$, the proof of Lemma 3.4 shows that the degree of η is the same as the degree of the map $\eta_0\colon M_0 \to U^{n-3} \times [0,1)$ defined by $\eta(c,t,0) = (\eta_0(c,t), 0)$, where $M_0 = \{(c,t)\colon (c,t,0) \in P_0\}$.

The map η_0 can be easily understood, as follows. Let $h\colon J \to U$ be a conformal homeomorphism that takes $\gamma(0)$ to 0. For $z \in U - \{0\}$, let $\lambda_z\colon U \to U$ be the rotation of U that takes z into $(0,1)$. Then

$$\eta_0(c,t) = \left(\lambda_{h(\gamma(t))}\left(h\left(c_1\right)\right), \ldots, \lambda_{h(\gamma(t))}\left(h\left(c_{n-3}\right)\right), \lambda_{h(\gamma(t))}\left(h\left(\gamma\left(t\right)\right)\right)\right),$$

wherever η_0 is defined. It is then quite easy to verify that η_0 has degree 1. (For example, one can use a proper homotopy from γ to the curve $h^{-1}([0,1))$ to get a proper homotopy from η_0 to the map $(c,t) \to (h(c_1), \ldots, h(c_{n-3}), t)$. Since h is an orientation preserving homeomorphism, this latter map has degree 1.) This shows that η is surjective.

Now suppose that $n > 2$, and let $A \in \mathcal{D}_J^n$. Let f_1 be some conformal automorphism of J such that $A_1 = f_1(A) \in \mathcal{D}_J^{n*}$, and let p be some point in $\eta^{-1}(\{C^{-1} \circ \psi(A_1)\})$. Then A is conformally homeomorphic to $\phi(p)$, and the conformal homeomorphism $g_{\phi(p)}^{-1} \circ g_{A_1} \circ f_1\colon A \to \phi(p)$ satisfies all the requirements. ∎

5. Periodic uniformization

We now state a theorem about doubly periodic uniformizations of doubly periodic domains.

5.1 Theorem: *Let $D \subsetneq \mathbb{C}$ be a doubly periodic domain; that is, D is invariant under two translations, $z \to z + \omega_1$ and $z \to z + \omega_2$, with ω_1, ω_2 linearly independent over $\mathbb{R}$. Let Γ be the group generated by these translations, and assume that D/Γ has finitely many boundary components in the torus $\mathbb{C}/\Gamma$, say $B_1, B_2, \ldots, B_n$. Let $K_1, \ldots, K_n \in \mathcal{H}$ be nondegenerate bounded holes. Then* *there is a conformal homeomorphism $f\colon D \to D^*$ of D onto a doubly periodic domain $D^* \subset \mathbb{C}$, such that $f(z+\omega_m) = f(z) + \omega_m^*$ holds for $m = 1,2$ and all $z \in D$, where ω_1^*, ω_2^* are periods of D^*, and such that for every $j = 1, \dots, n$ each boundary component $\hat{B}_j$ of D that maps to B_j under the projection $\mathbb{C} \to \mathbb{C}/\Gamma$ corresponds under f to a component of $\mathbb{C} - D^*$ that is homothetic to K_j or is a singleton.*

Furthermore, let $q_0 \neq q_1$ be two points in D, let A_0, A_1 be two distinct boundary components of D, and let i_0, i_1 be the indices such that A_j projects to B_{i_j} for $j = 0,1$. Suppose that $0 \in K_{i_j}$ for $j = 0,1$. Then f can also be required to satisfy any one of the following normalizations.

(1) *$f(q_0) = 0$ and $f(q_1) = 1$.*
(2) *$f(q_0) = 0$ and $f'(q_0) = 1$.*
(3) *For $j = 0,1$, the boundary component A_j^* of D^* that corresponds to A_j has the form $j + r_j K_{i_j}$ for some $r_j \geqslant 0$.*
(4) *$f(q_1) = 1$ and the boundary component A_0^* of D^* that corresponds to A_0 has the form rK_{i_0} for some $r \geqslant 0$.*

If we choose $q_1 = q_0 + \omega_1$ in normalization (1), we get $\omega_1^* = 1$.

Remark: Similar theorems hold for singly periodic domains. Actually, when D is singly periodic and D/Γ has finitely many boundary components in $\mathbb{C}/\Gamma$ (here Γ is the infinite cyclic group of translations that take D into itself), one can take advantage of the fact that the cylinder $\mathbb{C}/\Gamma$ is conformally homeomorphic to $\mathbb{C}^* = \mathbb{C} - \{0\}$ to get periodic uniformizations for D from the theorems of the previous sections. This is one example for the use of uniformizations as in Remark 3.2, where the boundary components are not specified up to homotheties — if one wishes to get a periodic uniformization for D with the boundary components specified up to homothety, one can uniformize D/Γ in $\mathbb{C}^*$ with boundary components specified as the images of a hole under homotheties followed by the exponential map.

We will need some definitions for the proof. Let $x^1, x^2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and suppose that H is a hole, if the union

$$\tilde{H} = \bigcup_{m_1, m_2 \in \mathbb{Z}} H + m_1 x^1 + m_2 x^2$$

is a disjoint union, then it is called an (x^1, x^2) periodic hole. Note that an (x^1, x^2) periodic hole is closed in $\mathbb{C}$, since x^1 and x^2 are linearly independent over $\mathbb{R}$.

We now define a space $\mathcal{D}_p^n$ of doubly periodic domains, as follows. A point $x \in \mathcal{D}_p^n$ has the form $x = (x^1, x^2, X^1, X^2, \dots, X^n)$, where $x^1, x^2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$, and $X^1, X^2, \dots, X^n$ are disjoint (x^1, x^2) periodic holes. The **domain underlying** x is $D(x) = \mathbb{C} - \bigcup_{j=1}^n X^j$. The set of $x \in \mathcal{D}_p^n$ such that $0, 1 \in D(x)$ will be denoted by $\mathcal{D}'$.

The topology on $\mathcal{D}_p^n$ is defined as follows. Let $x \in \mathcal{D}_p^n$, and let $\epsilon > 0$. Define $U(x, \epsilon)$ as the set of all $y \in \mathcal{D}_p^n$ such that $|y^1 - x^1| < \epsilon$, $|y^2 - x^2| < \epsilon$, $X^j \cap B(0, 1/\epsilon)$ does not intersect an unbounded component of $\mathbb{C} - N_\epsilon(Y^j)$ and $Y^j \cap B(0, 1/\epsilon)$ does not intersect an unbounded component of $\mathbb{C} - N_\epsilon(X^j)$ for $j = 1, 2, \dots, n$. Here $B(0, r)$ denotes the disk of radius r around 0, and $N_\epsilon(\cdot)$ denotes the ϵ-neighborhood of a set. The topology on $\mathcal{D}_p^n$ is defined as the collection of sets $W \subset \mathcal{D}_p^n$ such that for every $x \in W$ there is some $\epsilon > 0$ such that $U(x, \epsilon) \subset W$.

Below, we will make use of the following theorem from [12].

5.2 Theorem: *Let $D \subset \hat{\mathbb{C}}$ be a domain with countably many boundary components in $\hat{\mathbb{C}}$. Then there is a conformal homeomorphism g from D onto a domain $D^* \subset \hat{\mathbb{C}}$ whose boundary components are all circles and points. Moreover, g is unique up to post-composition by Möbius transformations.*

It is possible that our application of this theorem can be replaced by more classical results.

Proof of 5.1: We will only consider normalization (1); the other normalizations are treated similarly.

Let $x = (x^1, x^2, X^1, \dots, X^n) \in \mathcal{D}'$. Since the boundary of $D(x)$ has countably many boundary components, by 5.2 there is a conformal homeomorphism g_x from $D(x)$ onto a domain whose boundary components are circles and points, and g_x is unique up to post-composition by Möbius transformations. From the uniqueness we conclude that g_x conjugates the translations $z \to z + x^1, z \to z + x^2$ of D to Möbius transformations; that is

$$g_x(z + x^k) = M_k(g_x(z))$$

is valid for $k = 1, 2$, where M_1, M_2 are some Möbius transformations. The boundary component of $g_x(D(x))$ in $\hat{\mathbb{C}}$ that corresponds to the boundary component $\{\infty\}$ of $D(x)$ in $\hat{\mathbb{C}}$ is therefore stabilized by a discrete group of Möbius transformations that is isomorphic to $\mathbb{Z}^2$, and must therefore be a single point. We now make the normalization that this boundary component is $\{\infty\}$ and that

$g_x(0) = 0, g_x(1) = 1$. This normalization uniquely determines g_x. Moreover, with this normalization, the Möbius transformations M_1, M_2 must also be translations, say by y^1, y^2, respectively. Thus $y = (y^1, y^2, Y^1, \dots, Y^n)$ is in $\mathcal{D}'$, where Y^j is the periodic hole in $\mathbb{C} - g_x(D)$ that corresponds to X^j under g_x. Now define $\psi\colon \mathcal{D}' \to \mathcal{D}'$ by $\psi(x) = y$.

We shall demonstrate that ψ is continuous. Suppose that a sequence $x_1, x_2, \dots$ in $\mathcal{D}'$ converges to $x_0 \in \mathcal{D}'$. Then every compact subset of the domain $D(x_0)$ is contained in all but finitely many of the domains $D(x_k)$. From the normalization of the maps g_x, it is clear that the sequence g_{x_k} forms a normal family. We may assume therefore, without loss of generality, that these maps converge uniformly on compact subsets of $D(x_0)$, say to g. The function g cannot be a constant, since $g(0) = 0, g(1) = 1$, and g is therefore a conformal homeomorphism. Set $y_k = \psi(x_k)$. Since $y_k^1 = g_{x_k}(x_k^1)$, and similarly for y_k^2, the convergence of the sequence g_{x_k} implies that the sequences y_k^1, y_k^2 also converge, say to y^1, y^2. It is also clear that $g(D(x_0))$ is a doubly periodic domain with periods y^1, y^2. Thus the boundary component in $\hat{\mathbb{C}}$ of $g(D(x_0))$ that corresponds to the boundary component $\{\infty\}$ of $D(x_0)$ is $\{\infty\}$. Since all other boundary components of $D(x_0)$ are isolated, it is easy to establish with a limit argument that all the boundary components of $g(D(x_0))$ are circles and points. By the uniqueness of g_{x_0}, this then implies that $g = g_{x_0}$, and the continuity of ψ follows.

The proof that ψ is proper proceeds very much like the argument in the proof of Theorem 3.1, but more details need to be checked. Let $W \in \mathcal{D}^{n+1}$ be a bounded domain with $n+1$ labeled boundary components such that $0, 1 \in W$ and $\infty \in \hat{\partial}_{n+1} W$, and let $E_1, E_2 \subset W$ be compact sets such that e_1, e_2 are linearly independent over $\mathbb{R}$ whenever $e_1 \in E_1, e_2 \in E_2$. Define

$$\delta = \delta(E_1, E_2) = \inf\{|te_1 + se_2| \colon e_1 \in E_1, e_2 \in E_2,\ t, s \in \mathbb{R},\ t^2 + s^2 \geqslant 1\};$$

clearly, $\delta > 0$. Define $U(W, E_1, E_2) \subset \mathcal{D}'$ as the set of all $y = (y^1, y^2, Y^1, \dots, Y^n) \in \mathcal{D}'$ such that $W \subset D(y)$ and for each $j = 1, \dots, n$ the intersection $\hat{\partial}_j W \cap (\mathbb{C} - D(y))$ is a connected component of Y^j and y^k is in the interior of E_k for $k = 1, 2$. Clearly, $U(W, E_1, E_2)$ is open in $\mathcal{D}'$, and the open sets of this form cover $\mathcal{D}'$.

The collection $\mathcal{T}$ of all conformal maps h of W that take W into $\mathbb{C}$ and satisfy $h(0) = 0, h(1) = 1, \infty \in \hat{\partial}_{n+1} h(W)$ is compact under the topology of uniform convergence on compact subsets of W. This means that there is some $\epsilon > 0$

such that for every $h \in \mathcal{T}$ the distance between any two boundary components of $h(W)$ is at least ϵ, the ϵ neighborhood of any boundary component of $h(W)$ does not separate the other boundary components of $h(W)$, $\cup_{j=1}^{n} \hat{\partial}_j h(W) \cup h(E_1 \cup E_2)$ is contained in the disk $B(0, 1/\epsilon)$, and $h(W \cap B(0,\delta)) \supset B(0,\epsilon), h(W) \supset B(1,\epsilon)$.

Now suppose that $x \in \psi^{-1}(U(W, E_1, E_2))$. The restriction of the map g_x^{-1} to W is in $\mathcal{T}$. We therefore conclude that the distance between distinct boundary components of $D(x)$ is at least ϵ, the ϵ neighborhood of any boundary component of $D(x)$ does not separate the other boundary components of $D(x)$, the diameter of any boundary component of $D(x)$ is at most $1/\epsilon$, the distance from $\{0,1\}$ to the boundary of $D(x)$ is at least ϵ, and $x^1, x^2 \in B(0, 1/\epsilon)$. Moreover, since $g_x(m_1 x^1 + m_2 x^2) = m_1 g_x(x^1) + m_2 g_x(x^2)$ holds for $m_1, m_2 \in \mathbb{Z}$ and $g_x(x^1) \in E_1, g_x(x^2) \in E_2$, we conclude from the definition of δ and from $g_x(B(0,\epsilon) \cap D(x)) \subset B(0,\delta)$ that the infimum of $|m_1 x^1 + m_2 x^2|$ for $(0,0) \neq (m_1, m_2) \in \mathbb{Z}^2$ is greater than ϵ. Thus the closure of $\psi^{-1}(U(W, E_1, E_2))$ in $\mathcal{D}'$ is compact. Using this, it is easy to establish that ψ is proper, as in the proof of Theorem 3.1.

Let $M = \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{R}^2/\mathbb{Z}^2)^n$, where $\mathrm{GL}_2(\mathbb{R})$ is the group of nonsingular linear transformations of $\mathbb{R}^2$, and let $M_+ \subset M$ consist of all points of M whose projection to $\mathrm{GL}_2(\mathbb{R})$ has positive determinant. (We identify $\mathbb{C}$ with $\mathbb{R}^2$, and $\mathbb{Z}^2$ with the set $\{n_1 + n_2 i\colon n_1, n_2 \in \mathbb{Z}\}$.) For

$$p = (T, c_1 + \mathbb{Z}^2, c_2 + \mathbb{Z}^2, \ldots, c_n + \mathbb{Z}^2, r_1, r_2, \ldots, r_n) \in M \times \mathbb{R}_+^n$$

define $x(p) = (x^1(p), x^2(p), X^1(p), \ldots, X^n(p))$ by $x^1(p) = T(1), x^2(p) = T(i)$, and

$$X^j(p) = r_j K_j + T(c_j + \mathbb{Z}^2),$$

for $j = 1, 2, \ldots, n$. Let $P \subset M \times \mathbb{R}_+^n$ be the set of points $p \in M \times \mathbb{R}_+^n$ such that $x(p)$ is in $\mathcal{D}'$. Similarly, define $y(p)$ in the same way, but with each K_j replaced by a compact geometric disk, and let $Q \subset M \times \mathbb{R}_+^n$ be the set of all $p \in M \times \mathbb{R}_+^n$ such that $y(p) \in \mathcal{D}'$. It is clear that $x\colon P \to \mathcal{D}'$ and $y\colon Q \to \mathcal{D}'$ are continuous injective proper maps and that the image of y coincides with the image of ψ. Moreover, since y is injective, continuous, and proper and $\mathcal{D}'$ is locally compact, it follows that $y\colon Q \to y(Q)$ is a homeomorphism.

Let P_0 be the connected component of P that intersects $M_+ \times \{0\}$, and consider the map $\eta = y^{-1} \circ \psi \circ x\colon P_0 \to Q$. This map satisfies the requirements of the Degree Lemma 3.4, and is therefore surjective onto the connected component of

Q that contains $\eta(P_0)$. This connected component consists of all $q \in Q$ whose projection to $\mathrm{GL}_2(\mathbb{R})$ has positive determinant. The theorem with normalization (1) now easily follows. ∎

6. Packings

We now state our generalization of the Circle Packing Theorem.

6.1 PACKING THEOREM: *Let $G = (V, E)$ be a finite planar graph, let $a \in V$ be some vertex of the graph, and let $(P_v\colon v \in V)$ be a collection of smooth (C^1) closed topological disks in $\hat{\mathbb{C}}$ indexed by the vertices of G. Suppose that $\infty \in \text{interior}(P_a)$ and $\infty \notin P_v$ for all $v \in V - \{a\}$. Then there is a packing $Q = (Q_v\colon v \in V)$ whose contacts graph is G such that each Q_v is homothetic to P_v and $Q_a = P_a$.*

It is possible to require additional constraints on the packings that arise from the normalizations in the conformal uniformization. To describe these we need the following definition.

6.2 Definition: Let $Q = (Q_v\colon v \in V)$ be a packing in $\hat{\mathbb{C}}$, let w, u be distinct vertices in V, and let $p \in \hat{\mathbb{C}}$. We say that the sets Q_w and Q_u **cover** p (for the packing Q) if $p \in Q_w \cap Q_u$ or if p is in the closure of a connected component of $\hat{\mathbb{C}} - (Q_w \cup Q_u)$ that is disjoint from $\bigcup_{v\in V} Q_v$.

6.3 THEOREM: *In the situation of Theorem 6.1, let p_0, p_1, p_2 be distinct points on ∂P_a, and let $\gamma\colon [0,1) \to \hat{\mathbb{C}} - P_a$ be continuous and proper. Suppose that b_0, b_1, b_2 are three distinct neighbors of a in G, $c_0, c_1 \in V - \{a\}$ are neighbors in G, $d_0, d_1 \in V - \{a, b_0\}$, and d_0 is not a neighbor of a. Also assume that $0 \in P_v$ for $v \in \{d_0, d_1\}$. Then the packing in Theorem 6.1 can be further required to satisfy any one of the following normalizations.*

1. *Q_a and Q_{b_j} cover p_j for $j = 0, 1, 2$.*
2. *Q_a and Q_{b_0} cover p_0, and Q_{c_0} and Q_{c_1} cover $\gamma(0)$.*
3. *$Q_{d_0} = r_0 P_{d_0} + \gamma(0)$ and $Q_{d_1} = r_1 P_{d_1} + \gamma(t)$, for some $r_0, r_1 \in \mathbb{R}_+$ and $t \in [0,1)$.*
4. *Q_a and Q_{b_0} cover p_0, and $Q_{d_0} = r_0 P_{d_0} + \gamma(0)$ for some $r_0 \in \mathbb{R}_+$.*

These normalizations correspond to normalizations (1), (3), (5) and (6) in Theorem 4.2. It is left to the reader to formulate the analogs of (4) and (7) of 4.2.

As Figure 6.1 illustrates, it is essential that the definition of 'cover' include the possibility that the covered point is not in the intersection of the two sets.

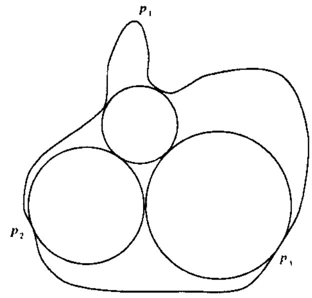

Figure 6.1.

Proof of 6.1 and 6.3: Assume without loss of generality that the graph G is isomorphic to (the 1-skeleton of) a triangulation of the sphere $\hat{\mathbb{C}}$. (For if G is not isomorphic to a triangulation, then new vertices and edges can be appended to G to make it a triangulation without inserting any new edges between two existing vertices. Then one chooses disks P_v to correspond to the new vertices.) It will be convenient not to distinguish between the abstract graph G and its geometric realization as a triangulation. Hopefully, this will cause no confusion.

We now make a somewhat arbitrary construction. Let B_a be some Jordan curve in $\hat{\mathbb{C}}$ that separates a from all the other vertices of G and intersects each edge incident with a exactly once, and let A be the Jordan domain bounded by B_a that does not contain a. On every edge e of G, choose some point $x_e \in e$ that is not a vertex, and for edges incident with a let that point be the point of intersection $e \cap B_a$. For $m = 1, 2, 3, \ldots$, $v \in V - \{a\}$ and e an edge containing v let $x_{e,v}(m)$ be a point in the subarc of e joining v to x_e, chosen so that $x_{e,v}(m) \neq x_e$ and $\lim_{m\to\infty} x_{e,v}(m) = x_e$, and let $B_{e,v}(m)$ be the closed subarc of e extending from v to $x_{e,v}(m)$. For $v \in V - \{a\}$, $m = 1, 2, 3, \ldots$, let $B_v(m) = \bigcup_e B_{e,v}(m)$, where the union extends over all edges e that contain v, and let $B_a(m) = B_a$. Finally, let $D(m)$ be the domain $A - \bigcup_{v\in V-\{a\}} B_v(m)$. See Figure 6.2.

Consider normalization 6.3.(2) first. Let e_0 be the edge that connects a and b_0, and let e_1 be the edge that connects c_0 and c_1. By Theorem 4.2 with normalization (3), it follows that for every $m = 1, 2, 3, \ldots$, there are disjoint sets $(Q_v(m)\colon v \in V)$ with each $Q_v(m)$ homothetic to P_v or a point and with $Q_a(m) = P_a$, such that $D(m)$ is conformally homeomorphic to $D^*(m) \overset{\text{def}}{=} \hat{\mathbb{C}} - \bigcup_{v\in V} Q_v(m)$. Furthermore, we assume, as we may, that (the continuous extension to $D(m) \cup B_a$

of) the conformal homeomorphism f_m: $D(m) \to D^*(m)$ satisfies $f_m(x_{e_0}) = p_0$ and $f_m(x_{e_1}) = \gamma(0)$. By choosing a subsequence, we also assume without loss of generality that $Q_v \stackrel{\text{def}}{=} \lim_{m\to\infty} Q_v(m)$ exists for every $v \in V$. We shall prove that $Q = (Q_v: v \in V)$ is a packing as required.

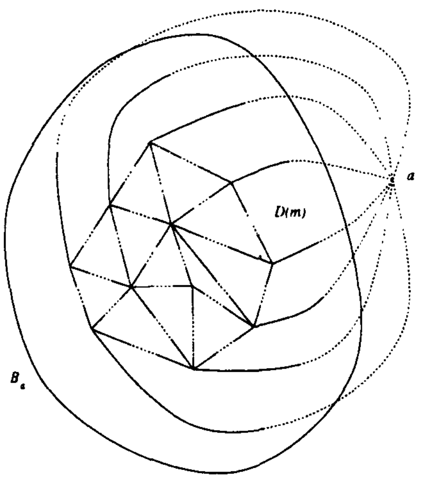

Figure 6.2. The domain $D(m)$.

First, it is clear that each Q_v is either homothetic to P_v or a point, and that $Q_a = P_a$. It is also obvious that each Q_v is disjoint from the interior of each $Q_w, w \in V - \{v\}$. Now consider two neighboring vertices v, w. Since the extremal length of the curve family in $D(m)$ joining $B_v(m)$ and $B_w(m)$ tends to zero as $m \to \infty$, it follows that the same is true for the curve family joining $Q_v(m)$ and $Q_w(m)$ in $D^*(m)$, and therefore, the distance between $Q_v(m)$ and $Q_w(m)$ tends to zero. (See [16] for the notion of extremal length and its properties.) In the limit, we see that Q_v and Q_w must intersect.

We shall now verify that Q_{c_0} and Q_{c_1} cover $\gamma(0)$. Let γ_0, γ_1 be simple curves that join c_0 and c_1, are disjoint from G, except at c_0 and c_1, and their union separates x_{e_1} from $V - \{c_0, c_1\}$. For each m, let $\Gamma_0(m)$ be the family of all curves in $D(m)$ that join $B_{c_0}(m)$ and $B_{c_1}(m)$ and that are homotopic to γ_0 in $D(m) - \{x_{e_1}\}$ with endpoints kept on $B_{c_0}(m)$ and $B_{c_1}(m)$. Let $\Gamma_1(m)$ be similarly defined from γ_1. The extremal length of $\Gamma_0(m)$ tends to zero as $m \to \infty$. Therefore, the same is true for the families $\Gamma_0^*(m) \stackrel{\text{def}}{=} \{f_m(\gamma): \gamma \in \Gamma_0(m)\}$. This means that the length of the shortest curve in $\Gamma_0^*(m)$ tends to zero. Since the same is true for $\Gamma_1^*(m) \stackrel{\text{def}}{=} \{f_m(\gamma): \gamma \in \Gamma_1(m)\}$, and since whenever $\alpha_0 \in \Gamma_0^*(m), \alpha_1 \in \Gamma_1^*(m)$ the union $\alpha_0 \cup \alpha_1 \cup Q_{c_0}(m) \cup Q_{c_1}(m)$ separates $\gamma(0) = f_m(x_{e_1})$ from each $Q_v(m)$, $v \neq c_0, c_1$, it follows that in the limit Q_{c_0} and Q_{c_1} cover $\gamma(0)$.

The proof that Q_a and Q_{b_0} cover p_0 is similar, and is left to the reader. Thus 6.3.(2) is verified.

We now show that none of the sets $Q_v, v \in V$ degenerates to a point. Let $V_0 \subset V$ be the collection of all vertices v such that Q_v is a point. With the intention of reaching a contradiction, we assume that $V_0 \neq \emptyset$. Let U_0 be some connected component of V_0 in G (in the graph theoretic sense). Since $Q_v \cap Q_w \neq \emptyset$ whenever v and w neighbor in G, it follows that $\bigcup_{v\in U_0} Q_v$ is a single point, say $\bigcup_{v\in U_0} Q_v = \{y\}$. Let ∂U_0 be the set of vertices that are not in U_0 but neighbor with a vertex in U_0.

∂U_0 can contain at most two vertices, because $y \in \bigcap_{v\in\partial U_0} Q_v$, and it is impossible for a collection of more than two smooth sets with disjoint interiors to have nonempty intersection. Since G is a triangulation of the sphere, it is 3-connected; that is, removing less than three vertices from G does not disconnect it. It therefore follows that $V = U_0 \cup \partial U_0$. But $a \notin V_0 \supset U_0$, so $a \in \partial U_0$. Since a, c_0, c_1 are all distinct and $|\partial U_0| < 3$, at least one of c_0, c_1 must be in U_0, say $c_0 \in U_0$. Because Q_{c_0} and Q_{c_1} cover $\gamma(0)$, we must have $Q_{c_0} = \{\gamma(0)\} = \{y\}$. That gives a contradiction, since $a \in \partial U_0$ and $\gamma(0) \notin Q_a$. This contradiction establishes that $V_0 = \emptyset$, and there are no degeneracies.

We now see that Q is a packing of smooth sets and its contacts graph contains G. Since the contacts graph must be planar, and since G is a triangulation, it follows that the contacts graph is G. This completes the proof with normalization 6.3.(2).

In the proof of 6.3 with normalization 6.3.(3) we proceed similarly, but use normalization (5) of Theorem 4.2 in place of 4.2.(3). This ensures that 6.3.(3) holds. The proof that no sets in $Q = (Q_v \colon v \in V)$ degenerate to points needs some modifications for this case. Again, let V_0 be the set of vertices v such that Q_v is a singleton, and assume that $V_0 \neq \emptyset$. Let U_0 be a connected component of V_0. As before, we see that $|\partial U_0| \leqslant 2$, and conclude that $V = U_0 \cup \partial U_0$. Let $\{y\} = \bigcup_{v\in U_0} Q_v$, then $y \in \partial Q_a$, since $a \notin U_0$. This then implies that $d_0 \notin U_0$, because $\gamma(0) \in Q_{d_0}$. So we have $V - V_0 = \partial U_0 = \{a, d_0\}$. Let $v_0, v_1, v_2, \ldots, v_k$ be the neighbors of d_0 in cyclic order, with $v_k = v_0$. For m large there is a sequence of arbitrarily short curves $\beta_1, \ldots, \beta_k$ such that each β_j joins $Q_{v_{j-1}}(m)$ and $Q_{v_j}(m)$ and $\bigcup_{j=1}^{k}(\beta_j \cup Q_{v_j}(m))$ surrounds $Q_{d_0}(m)$. Since the diameter of each $Q_{v_j}(m)$

($j = 1, \ldots, k$) tends to zero as $m \to \infty$, this implies that the diameter of $Q_{d_0}(m)$ tends to zero. But this contradicts the fact that $d_0 \notin U_0$. Thus the proof with normalization 6.3.(3) is complete.

The proofs for normalizations (2) and (4) are similar, and are left to the reader. ∎

6.4 Remark: We have not attempted for maximal generality in the Packing Theorem 6.1. The packed sets do not have to be prescribed by homotheties; there are many other possible variations. See, e.g., Remark 3.2. Also, as the proof shows, the smoothness requirement can be replaced by any condition that will insure that the intersection of more than two packed sets is empty. For example, one can require that the packed sets do not have angles $\leqslant 2\pi/3$. If the smoothness requirement is dropped without replacement, then Theorems 6.1 and 6.3 remain intact, except for two possible types of degeneracies: the contacts graph of the packing will contain G, it may also contain additional edges; and some sets in the packing may degenerate to points.

We now prove a theorem about doubly periodic packings.

6.5 THEOREM: *Let $\pi: \mathbb{C} \to T$ be the universal cover of a torus T, and let τ_1, τ_2 be generators for the group of deck transformations Γ. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a Γ-invariant graph embedded in $\mathbb{C}$, and suppose the the projection $G = G(V, E) = \pi(\tilde{G})$ is a finite graph. (G may have loops and multiple edges.) For each $v \in V$ let P_v be some smooth (C^1) compact topological disk in $\mathbb{C}$. Suppose that $a, b \in \tilde{V}$ are distinct, and assume that $0 \in P_a, 0 \in P_b$.*

Then there exists a packing $Q = (Q_v: v \in \tilde{V})$ in $\mathbb{C}$ whose contacts graph is $\tilde{G}$ such that each Q_v is homothetic to $P_{\pi(v)}$ and $Q_a = r_1 P_{\pi(a)}$, $Q_b = 1 + r_2 P_{\pi(b)}$ hold for some $r_1, r_2 > 0$. The packing can also be required to be doubly periodic; that is, there are ω_1, ω_2 such that $Q_{\tau_j(v)} = \omega_j + Q_v$ holds for every $v \in \tilde{V}$ and $j = 1, 2$.

In the situation where all the sets P_v are strictly convex and $\tilde{G}$ is the 1-skeleton of a triangulation of $\mathbb{C}$, it is not known if there can be a packing Q that fails to be doubly periodic but does satisfy all the other requirements. This cannot happen if all the P_v are disks. On the other hand, the "bricks" tiling $([2m+n, 2m+n+2] \times [2n, 2n+2]: n, m \in \mathbb{Z})$ can be deformed to a non-doubly periodic tiling $([2m+s(n), 2m+s(n)+2] \times [2n, 2n+2]: n, m \in \mathbb{Z})$. This shows that there is a packing with doubly periodic triangulation graph that is not doubly periodic if one replaces the strict convexity by convexity (the corners may be rounded so that the packed sets are smooth). This question is related to the problems of uniqueness, which will be addressed in the next section.

Proof: The proof is similar, of course, to the proof of Theorems 6.1 and 6.3. Assume, without loss of generality, that $\tilde{G}$ is the 1-skeleton of a triangulation of $\mathbb{C}$. On every edge e of $\tilde{G}$, choose some point $x_e \in e$ that is not a vertex. For $m = 1, 2, 3, \ldots$, $v \in \tilde{V}$ and e an edge containing v, let $x_{e,v}(m)$ be a point in the subarc of e joining v to x_e, chosen so that $x_{e,v}(m) \neq x_e$ and $\lim_{m\to\infty} x_{e,v}(m) = x_e$, and let $B_v(m)$ be the union of the arcs extending from v to $x_{e,v}(m)$ along all edges e that contain v. Let $D(m)$ be the domain $\mathbb{C} - \bigcup_{v\in\tilde{V}} B_v(m)$. The points $x_{e,v}(m)$ can be chosen Γ-periodically, and so we assume, without loss of generality, that $D(m)$ is Γ-periodic.

By Theorem 5.1, for every $m = 1, 2, \ldots$, there are disjoint sets $(Q_v(m)\colon v \in \tilde{V})$ such that each $Q_v(m)$ is homothetic to $P_{\pi(v)}$ or is a point, $Q_a(m) = r_1(m)P_{\pi(a)}$, $Q_b(m) = 1 + r_2(m)P_{\pi(b)}$ for some $r_1(m), r_2(m) \geqslant 0$, and there is a conformal homeomorphism $f_m\colon D(m) \to \mathbb{C} - \bigcup_{v\in\tilde{V}} Q_v$. Moreover, $(Q_v(m)\colon v \in \tilde{V})$ is periodic; that is, there are $\omega_1(m), \omega_2(m) \in \mathbb{C}$ linearly independent over $\mathbb{R}$ such that $Q_{\tau_j(v)}(m) = \omega_j(m) + Q_v(m)$ holds for $j = 1, 2$, $m = 1, 2, 3, \ldots$, and $v \in \tilde{V}$. Let $d(m)$ be the largest diameter of any of the sets $Q_v(m)$, and define the sets $Q'_v(m)$ by $Q'_v(m) = d(m)^{-1}Q_v(m)$. It is clear that for some subsequence all the sets $Q'_v(m)$ converge in the Hausdorff metric; let $Q' = (Q'(v)\colon v \in \tilde{V})$ be the limit configuration for such a subsequence.

Clearly, the configuration Q' is doubly periodic and the largest diameter of a set in Q' is 1. Moreover, the same extremal length argument as in the proof of 6.1 shows that $Q'_v \cap Q'_u \neq \emptyset$ whenever v, u are neighbors in $\tilde{G}$. We now check that none of the sets Q'_v is a singleton. Let V_0 be the set of $v \in \tilde{V}$ such that Q'_v is a singleton. If U_0 is a connected component of V_0, then, as before, $|\partial U_0| \leqslant 2$ and therefore, by 3-connectivity, $\tilde{V} = U_0 \cup \partial U_0$. But these assertions together contradict periodicity. This contradiction establishes that Q'_v is never a singleton.

It is now clear that for an appropriate $d > 0$ the packing $Q_v = dQ'_v$, $v \in \tilde{V}$ satisfies all the requirements. ∎

7. A discussion of uniqueness

In general, uniqueness does not hold in the uniformization and packing theorems above. Consider, for example, the situation in Theorem 1.1 with normalization

(1.1). When the sets K_j are such that uniqueness holds for each $D \in \mathcal{D}^n_{(\infty)}$, one gets a bijective correspondence between the space $\mathcal{S}$ of horizontal slit domains in $\mathcal{D}^n_{(\infty)}$ and the space $\mathcal{D}(K_1, \ldots, K_n)$ of domains in $\mathcal{D}^n_{(\infty)}$ of the form $\hat{\mathbb{C}} - \bigcup_{j=1}^n K_j^*$ with each K_j^* homothetic to K_j or a singleton. This bijective correspondence is then easily seen to be a homeomorphism. Therefore, uniqueness must fail when the space $\mathcal{D}(K_1, \ldots, K_n)$ is disconnected. Figure 7.1 displays such an example with $n = 2$.

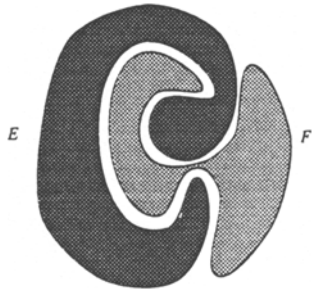

Figure 7.1. $\mathcal{D}(E, F)$ is disconnected.

On the other hand, under certain assumptions on the sets K_i, uniqueness can sometimes be proven. For example, as we have mentioned, Shiffman [23] has shown that in the special case where the sets K_j in Theorem 1.1 are all strictly convex (and one imposes the normalization (1.1)), the domain D^* and the conformal homeomorphism f are unique. The argument is based on counting the fixed points of $f_1 \circ f_2^{-1}(z)$, where f_1, f_2 are two hypothetical uniformizing homeomorphisms for D. The same approach is also applied in [12] to circle packings and uniformization by circle domains.

We now adapt this technique to packings with convex shapes (without smoothness or strict convexity assumptions).

7.1 Theorem: *Let $Q = (Q_v\colon v \in V)$, $P = (P_v\colon v \in V)$ be two packings in $\mathbb{C}$ of compact convex sets with nonempty interior. Suppose that for each $v \in V$ the two sets Q_v, P_v are homothetic, that the contacts graph G of P is also the contacts graph of Q, and that it is the 1-skeleton of a triangulation T of $\hat{\mathbb{C}}$. Let $[a, b, c]$ be a triangle in T, and assume that $Q_v = P_v$ for $v = a, b, c$. Further assume that all the sets Q_v, P_v are disjoint from the unbounded connected component of $\mathbb{C} - (Q_a \cup Q_b \cup Q_c)$. Then $Q = P$; the two packings are the same.*

Proof: Assume, as we may, that $V \neq \{a, b, c\}$. Denote the vertices, edges and triangles of T by V, E, F, respectively. Note that each of the packings Q, P induces an orientation on T, and these are in fact the same orientation (since they agree in the triangle $[a, b, c]$). Take T with that orientation.

We now construct a piecewise flat surface homeomorphic to $\hat{\mathbb{C}}$, as follows. Let T^* be a cell division of $\hat{\mathbb{C}}$ that is dual to T. Every vertex in T^* has three neighbors. Replace every vertex v in T^* by a triangle, and connect every vertex of this new triangle to a vertex in a new triangle corresponding a neighbor of v, as in Figure 7.2. Denote the resulting cell division by T'. We endow T' with a piecewise flat metric by declaring each face of T' to be a regular polygon with unit length edges.

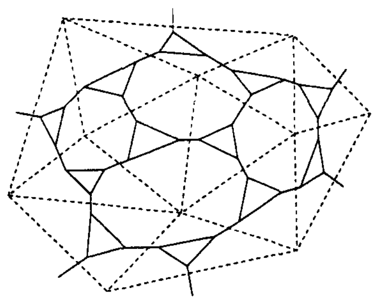

Figure 7.2. The cell division T'. The edges of T are indicated by broken lines.

Each triangle $t = [v_1, v_2, v_3]$ of T corresponds to a triangle s_t of T', each vertex v_j of t corresponds to an edge s_{t,v_j} of s_t and each edge of t corresponds to a vertex of s_t. Each edge e of T corresponds to an edge s_e of T'. Again, this is a duality correspondence, the edge s_e connects vertices in the triangles corresponding to faces of T containing e and lies on the boundary of polygons of T' corresponding to the vertices of e. Each vertex $v \in V$ corresponds to a $2n$-gon s_v in T', where n is the degree of v, the number of neighbors it has. There is one edge $s_{v,e}$ of s_v corresponding to each edge e that contains v, and there is one edge $s_{v,t}$ of s_v corresponding to each triangle t that contains v.

We now construct maps $f, g\colon H \to \hat{\mathbb{C}}$, where $H = \bigcup_{j \in E \cup F} s_j$. Consider a triangle $t = [v_1, v_2, v_3] \in F$. There are two possibilities: either the intersection $Q_{v_1} \cap Q_{v_2} \cap Q_{v_3}$ consists of a single point, p_t, or it is empty and there is a unique component k_t of $\hat{\mathbb{C}} - (Q_{v_1} \cup Q_{v_2} \cup Q_{v_3})$ that does not intersect any sets of the packing Q, and ∂k_t consists of three arcs, one on each Q_{v_j}, $j = 1, 2, 3$. In the first case, define f to be equal to p_t on s_t, and in the second, let $f|_{s_t}$ equal (the continuous extension of) the conformal map from s_t to k_t that takes each edge

s_{t,v_j} of s_t to $\partial k_t \cap Q_{v_j}$, $j = 1, 2, 3$. This defines f on $\bigcup_{t \in F} s_t$.

Now consider an edge $e = [v_1, v_2] \in E$. Note that f is already defined on the endpoints of s_e, since these are vertices of s_{t_1} and s_{t_2}, where t_1, t_2 are the two faces of T that contain e. The intersection $Q_{v_1} \cap Q_{v_2}$ consists of a single point, or a line segment. In each case, let f be affine on s_e. Then f maps s_e onto $Q_{v_1} \cap Q_{v_2}$. This completes the definition of f on H.

In exactly the same manner, but using the packing P instead of Q, we define the map $g\colon H \to \hat{\mathbb{C}}$. Let $v \in V$ be any vertex. Note that the restriction of f to ∂s_v traverses ∂Q_v weakly preserving orientation. That is, if p_1, p_2, p_3 are distinct points in positive circular order on ∂s_v and $f(p_1), f(p_2), f(p_3)$ are distinct, then they are in positive circular order on ∂Q_v. Similarly, g is weakly orientation preserving on every ∂s_v.

Suppose now that $P \neq Q$. Then there must be some $v_0 \in V$ such that P_{v_0} is not a translate of Q_{v_0}. Since P_{v_0} is homothetic to Q_{v_0}, there is a nonempty open set A of $\tau \in \mathbb{C}$ such that P_{v_0} is contained in the interior of $Q_{v_0} + \tau$ or $Q_{v_0} + \tau$ is contained in the interior of P_{v_0}.

Let $\tau \in A$. Suppose for the moment that for each $v \in V$ the intersection $\partial P_v \cap (\partial Q_v + \tau)$ contains at most two points and that $f - g$ does not attain the value τ on ∂H.

7.2 LEMMA: *Let $E_1, E_2 \subset \mathbb{C}$ be two Jordan curves oriented positively with respect to the domains they bound. Suppose that the intersection $E_1 \cap E_2$ contains at most two points. Let $f_1\colon S^1 \to E_1$, $f_2\colon S^1 \to E_2$ be continuous mappings that weakly preserve orientations, and suppose that $f_1 - f_2$ is never zero. Then the winding number of $f_1 - f_2\colon S^1 \to \mathbb{C}$ around 0 is nonnegative. If E_1 is contained in the domain bounded by E_2, then this winding number is 1.*

The proof is almost the same as the proof of the Circle Index Lemma (2.2) of [12]. See also [23].

Set $h(z) = f(z) - g(z) - \tau$. Since we are assuming that $|\partial P_v \cap (\partial Q_v + \tau)| \leqslant 2$, we can apply the above lemma with $f_1 = f|_{\partial s_v}$ and $f_2 = (g + \tau)|_{\partial s_v}$ and conclude that the winding number of h around ∂s_v is nonnegative for each $v \in V$. On the other hand, around ∂s_{v_0} the winding number of h is 1, again by the lemma. Let $t \in F, t \neq [a, b, c]$, then $\infty \notin f(s_t), \infty \notin g(s_t)$, and h is analytic in s_t. Therefore, the winding number around 0 of $h|_{\partial s_t}$ is nonnegative. For $t = [a, b, c]$, $h|_{\partial s_t}$ is a constant, by the construction and the assumption $P_v = Q_v$ for $v = a, b, c$.

Therefore, in this case the winding number is zero. So we get

$$\sum_{j \in V \cup F} \text{winding}(h|_{\partial s_j}, 0) \geqslant 1.$$

This is impossible, since each edge of T' is traversed twice in opposite directions in the curves ∂s_j $(j \in V \cup F)$.

It thus only remains to be seen that $\tau \in A$ can be chosen outside of the range of $(f-g)|_{\partial H}$ and such that $|P_v \cap (Q_v + \tau)| \leqslant 2$ for every $v \in V$. The image of ∂H under $f - g$ is easily seen to be a closed set with empty interior. Therefore the following lemma completes the proof of the theorem.

7.3 LEMMA: *Let $E_1, E_2 \subset \mathbb{C}$ be homothetic convex curves, and let B be the set of $\tau \in \mathbb{C}$ such that the intersection $E_1 \cap (E_2 + \tau)$ contains more than two points. Then B is of first Baire category. If E_1 is strictly convex, then B is empty.*

Proof: Let $\tau \in B$, and let $k(z) = \alpha z + \beta$, $\alpha > 0$ be the homothety that takes E_1 to $E_2 + \tau$. We will show that k takes some line whose intersection with E_1 is a nontrivial segment into a parallel line (possibly the same) with that property. Since there are at most countably many such lines, this will prove the lemma.

We assume that $k(z) = \alpha z$. There is no loss of generality in this assumption, since it can be achieved by a change of coordinates. Let y_1, y_2, y_3 be three distinct points in the intersection $E_1 \cap k(E_1)$, and set $x_j = \alpha^{-1} y_j = k^{-1}(y_j)$, $j = 1, 2, 3$. Then the six points $x_1, x_2, x_3, y_1, y_2, y_3$ are in E_1. If x_1, x_2, x_3 are collinear, then the line containing them intersects E_1 in a nontrivial segment, by the convexity of E_1, and k takes this line to the line containing y_1, y_2, y_3. Therefore, we may assume that x_1, x_2, x_3 are not collinear.

We analyze the different possibilities for the location of 0, the fixed point of k, in relation to the points x_1, x_2, x_3. See Figure 7.3.

If 0 is in the open regions D_1 or D_2 of Figure 7.3, then x_1 is in the interior of the convex hull of y_1, x_2, x_3. A contradiction to the convexity of E_1. If 0 is in the region D_3 of the figure, then y_1 is in the interior of the convex hull of x_1, y_2, y_3, a contradiction again. If 0 is on the line connecting x_2, x_3, then this line is invariant under k and must contain a nontrivial segment contained in E_1. By symmetry, these are the only cases we need to consider. This completes the proof. ∎

With the additional assumption that the sets Q_v are smooth and strictly convex, Theorem 7.1 is a special case of the results in [20]. The method of [20] and

[21, Theorems 5.3, 5.4], is a related topological technique for proving uniqueness of packings. For example, the following can be obtained from the means of [20].

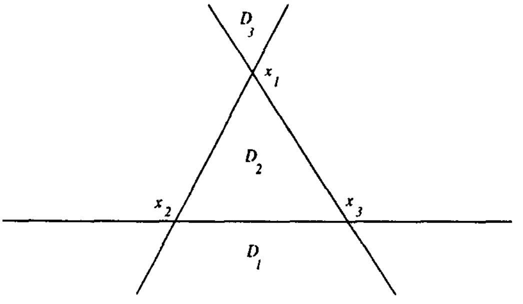

Figure 7.3. Possible locations of the fixed point of k.

7.4 THEOREM: *In Theorem 6.3, if G is isomorphic to the* 1*-skeleton of a triangulation of the sphere and the sets $P_v, v \in V - \{a\}$ are strictly convex, then normalization* (1) *uniquely determines the packing Q.*

Figure 7.4 gives an example of a situation where uniqueness fails for a triangulation graph and yet the sets $P_v, v \in V - \{a\}$ as well as $\hat{\mathbb{C}} - P_a$ are convex.

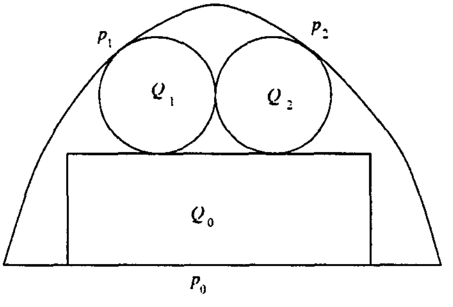

Figure 7.4. Nonuniqueness — slide Q_0 sideways.

References

[1] E. M. Andreev, *On convex polyhedra in Lobačevskiĭ spaces*, Matematicheskii Sbornik (N.S.) **81 (123)** (1970), 445–478 (Russian); Mathematics of the USSR-Sbornik **10** (1970), 413–440 (English).

[2] E. M. Andreev, *On convex polyhedra of finite volume in Lobačevskiĭ space*, Matematicheskii Sbornik (N.S.) **83 (125)** (1970), 256–260 (Russian); Mathematics of the USSR-Sbornik **12** (1970), 255–259 (English).

[3] M. Brandt, *Ein Abbildungssatz für endlich-vielfach zusammenhängende Gebiete*, Bulletin de la Société des Sciences et des Lettres de Łódź **XXX, 3** (1980).

[4] A. F. Beardon and K. Stephenson, *The Schwarz–Pick lemma for circle packings*, Illinois Journal of Mathematics **141** (1991), 577–606.

[5] I. Carter and B. Rodin, *An inverse problem for circle packing and conformal mapping*, Transactions of the American Mathematical Society **334** (1992), 861–875.

[6] Y. Colin De Verdiére, *Un principe variationnel pour les empilements de cercles*, Inventiones mathematicae **104** (1991), 655–669.

[7] R. Courant, B. Manel and M. Shiffman, *A general theorem on conformal mappings of multiply connected domains*, Proceedings of the National Academy of Sciences of the United States of America **26** (1940), 503–507.

[8] A. Dold, *Lectures on Algebraic Topology*, Springer, Berlin, 1972, p. 377.

[9] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, R.I., 1969, p. 676.

[10] V. Guillemin and A. Pollack, *Differential Topology*, Prentice Hall, Englewood Cliffs, NJ, 1974, p. 222.

[11] A. N. Harrington, *Conformal mappings onto domains with arbitrarily specified boundary shapes*, Journal d'Analyse Mathématique **41** (1982), 39–53.

[12] Z.-X. He and O. Schramm, *Fixed points, Koebe uniformization and circle packings*, Annals of Mathematics **137** (1993), 369–406.

[13] Z.-X. He and O. Schramm, *Koebe uniformization for 'almost circle domains'*, American Journal of Mathematics, to appear.

[14] P. Koebe, *Uber die Uniformisierung beliebiger analytischer Kurven III*, Nachrichten Gesellschaft für Wissenschaften in Göttingen (1908), 337–358.

[15] P. Koebe, *Kontaktprobleme der konformen Abbildung*, Berichte Verhande. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Klasse **88** (1936), 141–164.

[16] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, Berlin, 1973, p. 258.

[17] J. W. Milnor, *Topology from the Differential Viewpoint*, The University Press of Virginia, 1965, p. 64.

[18] B. Rodin and D. Sullivan, *The convergence of circle packings to the Riemann mapping*, Journal of Differential Geometry **26** (1987), 349–360.

[19] O. Schramm, *Packing two-dimensional bodies with prescribed combinatorics and applications to the construction of conformal and quasiconformal mappings*, Ph.D. thesis, Princeton, 1990.

[20] O. Schramm, *Existence and uniqueness of packings with specified combinatorics,* Israel Journal of Mathematics **73** (1991), 321-341.

[21] O. Schramm, *Rigidity of infinite (circle) packings,* American Journal of Mathematics **4** (1991), 127–149.

[22] O. Schramm, *How to cage an egg*, Inventiones mathematicae **107** (1992), 543–560.

[23] M. Shiffman, *Uniqueness theorems for conformal mappings of multiply connected domains*, Proceedings of the National Academy of Sciences of the United States of America **27** (1941), 137–139.

[24] R. J. Sibner, *'Uniformizations' of infinitely connected domains,* in *Advances in the Theory of Riemann Surfaces*, Proceedings of the 1969 Stony Brook Conference, Annals of Mathematics Studies, Princeton University Press, 1971, pp. 407–419.

[25] W. P. Thurston, *The Geometry and Topology of 3-Manifolds,* Princeton University Notes, Princeton, New Jersey, 1982.

[26] W. P. Thurston, *The finite Riemann mapping theorem,* invited address, International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, 1985.